Dominant Pole Region Assignment with Continuous PI and PID Controllers

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Abstract

This paper presents a method to design PI and PID controllers in continuous time domain through dominant pole region assignment so that the closed-loop time domain characteristics remain within the desired interval. Firstly, parametrization of the PI controllers which assign the dominant pole pair to the desired region is given. The sub-region, in which the remaining poles are located away from the dominant pole region, is then found in (K_p, K_i) parameter space by calculating the root boundaries. After that the method is extended for PID controller case by gridding the K_p parameter. Thus, all PID controllers, which perform dominant pole region assignment successfully, are given in 3D parameter space. Examples are given for two different pole regions and for both PI and PID controllers.

1. Introduction

Proportional integral derivative (PID) controllers and its variants are still the most popular controllers particularly in the industrial applications due to their simple structure, satisfactory performance and acceptable robustness [1, 2]. Because of the importance of PID controllers, various design methods have been considered in the literature [3-6]. However, the number of parameters to be tuned is only 2 for the PI controllers and 3 for the PID controllers; therefore, arbitrary assign of the all roots of closed-loop characteristic polynomial is not always possible especially for higher order systems. Nevertheless, the dominant pole placement approach, which is widely used in controller design for the linear time-invariant (LTI) systems to obtain the desired transient response using feedback controllers [7], can constitute a solution to the mentioned design problem encountered with PI/PID controllers.

It is known that the closed-loop poles are strongly responsible for the transient response of system. Therefore, a pair of dominant poles is assigned to the corresponding locations in order to provide the desired time domain characteristics such as settling time and overshoot [8]. Thus, the desired behavior in the closed-loop is obtained by only assigning two closed-loop poles. However, the adopted assumption here is that the remaining poles are located far away (3-5 times in general) from the dominant pole pair [9]. Although the dominant pole placement is an effective design method, if the assumption is violated (i.e. the remaining poles are not located far enough from the dominant poles), the desired performance specifications in the closed-loop are not guaranteed to be met [10].

Consider the fact that the degree of freedom for the PID controllers is enough to assign dominant poles but it is a challenge to keep the unassigned poles away from the dominant poles with only one remaining parameter (e.g. K_p). Furthermore, there is not any parameter left to change the locations of the remaining poles in case of the PI controller. Here, it is possible to widen the closed-loop performance specifications instead of choosing strict criteria. It may then become possible to find controller parameters with the help of this approach such that dominant pole placement is performed successfully. Besides, it is already meaningful for most of the systems to have time domain characteristics between the minimum and maximum desired values. This results the dominant pole pair to be located in a specified region instead of a point, hence, the dominant pole region assignment problem shows up.

In this study, it is aimed to find PI and PID controllers which assign two of the closed-loop poles in a desired region to satisfy some performance criteria whereas the remaining poles are located away from the dominant poles. The rest of paper is organized as follows: In the section 2, solution to the dominant pole region assignment problem is given for the PI controller first. After that the same approach is extended to the PID controller case. In the third section, the proposed method is demonstrated on example transfer functions both for PI and PID controllers, respectively. In the examples, two different pole regions are considered. Finally, conclusive remarks are given in the last section.

2. Dominant Pole Region Assignment

2.1. Pole Region Assignment with PI Controllers

Consider a SISO closed-loop control system depicted in Fig. 1 where

$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad (1)$$

with the continuous time PI controller,

$$F(s) = \frac{N_F(s)}{D_F(s)} = \frac{K_p\, s + K_i}{s} \qquad (2)$$

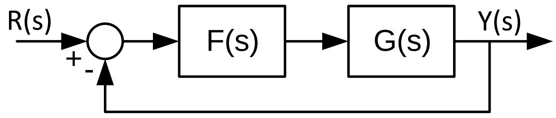

Fig. 1. A closed-loop control system.

Here, the problem is to find the PI controller parameters in $K_p - K_i$ plane such that the dominant poles are located in the desired region in s-plane whereas the remaining poles are located "m" times away from the dominant pole pair.

The closed-loop characteristic polynomial can be written as follows.

$$P_c(s) = D_F(s)D_G(s) + N_F(s)N_G(s) = 0 \tag{3}$$

Let the location of dominant pole pair in the closed-loop be expressed by $s_{1,2} = -\sigma \pm j\omega$. If the dominant pole locations are substituted into characteristic polynomial, we have,

$$D_F(\sigma,\omega)D_G(\sigma,\omega) + N_F(\sigma,\omega)N_G(\sigma,\omega) = 0 \tag{4}$$

This describes a complex equation that can be solved by decomposing into its real and imaginary parts,

$$\left(D_{F_{Im}} D_{G_{Im}} - D_{F_{Re}} D_{G_{Re}}\right) + \left(N_{F_{Im}} N_{G_{Im}} - N_{F_{Re}} N_{G_{Re}}\right) = 0 \tag{5}$$

$$\left(D_{F_{Re}} D_{G_{Im}} + D_{F_{Im}} D_{G_{Re}}\right) + \left(N_{F_{Re}} N_{G_{Im}} + N_{F_{Im}} N_{G_{Re}}\right) = 0 \tag{6}$$

where

$$D_{F_{Im}} = Im[D_F(-\sigma + j\omega)]\ ,\ D_{F_{Re}} = Re[D_F(-\sigma + j\omega)]$$
$$D_{G_{Im}} = Im[D_G(-\sigma + j\omega)]\ ,\ D_{G_{Re}} = Re[D_G(-\sigma + j\omega)]$$
$$N_{F_{Im}} = Im[N_F(-\sigma + j\omega)]\ ,\ N_{F_{Re}} = Re[N_F(-\sigma + j\omega)]$$
$$N_{G_{Im}} = Im[N_G(-\sigma + j\omega)]\ ,\ N_{G_{Re}} = Re[N_G(-\sigma + j\omega)]$$

Let us to make following definitions.

$$X = D_{F_{Im}} D_{G_{Im}} - D_{F_{Re}} D_{G_{Re}} \tag{7}$$
$$Y = D_{F_{Re}} D_{G_{Im}} + D_{F_{Im}} D_{G_{Re}} \tag{8}$$
$$Z = N_{G_{Im}}^2 + N_{G_{Re}}^2 \tag{9}$$

Furthermore, it is possible to write the followings for a PI controller.

$$N_{F_{Re}} = K_i - K_p\sigma \tag{10}$$
$$N_{F_{Im}} = K_p\omega \tag{11}$$

As a result, with the help of above expressions, PI controller parameters (K_p and K_i) can be obtained in terms of the parameters σ and ω (i.e. location of the dominant poles) as below.

$$K_p(\sigma,\omega) = -\frac{N_{G_{Im}}X + N_{G_{Re}}Y}{\omega\, Z} \tag{12}$$

$$K_i(\sigma,\omega) = -\frac{N_{G_{Im}}Y - N_{G_{Re}}X}{Z} + K_p(\sigma,\omega)\,\sigma \tag{13}$$

Thus, the parametrization of the PI controllers that assign two of the closed-loop poles to the points of $s_{1,2} = -\sigma \pm j\omega$ is completed.

Consider the following region, in which the dominant pole pair is desired to be placed,

$$\mathcal{D}_1 = \left\{ \begin{matrix} s = -\sigma \pm j\omega \ \mid\ \sigma,\omega \in \mathbb{R}^+, \\ \sigma_{min} \le \sigma \le \sigma_{max},\ \ \omega_{min} \le \omega \le \omega_{max} \end{matrix} \right\} \tag{14}$$

It is possible to map the $\mathcal{D}$ region to the PI controller parameter space by plotting the functions $K_p(\sigma,\omega)$ and $K_i(\sigma,\omega)$ for the given intervals of σ and ω.

It is worth to note that the dominant pole region can also be bounded by the different performance specifications such as settling time, damping ratio (ζ) or natural frequency (ω_n). Therefore, the parametrization given by (12) and (13) can also be obtained in terms of the other parameters. For instance, by substituting $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ in (3) for the following region

$$\mathcal{D}_2 = \left\{ \begin{matrix} s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \ \mid\ \zeta,\omega_n \in \mathbb{R}^+, \\ \zeta_{min} \le \zeta \le \zeta_{max},\ \ \omega_{n_{min}} \le \omega_n \le \omega_{n_{max}} \end{matrix} \right\} \tag{15}$$

or $s_{1,2} = -\sigma \pm j\frac{\sigma}{\zeta}\sqrt{1-\zeta^2}$ for the region given below.

$$\mathcal{D}_3 = \left\{ \begin{matrix} s = -\sigma \pm j\dfrac{\sigma}{\zeta}\sqrt{1-\zeta^2} \ \mid\ \zeta,\sigma \in \mathbb{R}^+, \\ \zeta_{min} \le \zeta \le \zeta_{max},\ \ \sigma_{min} \le \sigma \le \sigma_{max} \end{matrix} \right\} \tag{16}$$

It is clear that two of the closed-loop poles are located in the desired region as long as (K_p, K_i) pair is chosen from the obtained controller parameter space; however, it is required to find the sub-region in which the remaining closed-loop poles are also located "m" times away from the dominant pole pair.

$$\widetilde{\mathcal{D}} = \{s \in \mathbb{C} \mid Re(s) \le -m\sigma,\ \ \sigma, m \in \mathbb{R}^+\} \tag{17}$$

The closed-loop characteristic polynomial with the PI controller, whose parameters are obtained in terms of the σ and ω for the region given in (14), can be written as follows.

$$P_c(s,\sigma,\omega) = (s^2 + 2\sigma s + \sigma^2 + \omega^2)\, P_e(s,\sigma,\omega) \tag{18}$$

where $P_e(s,\sigma,\omega)$ is the polynomial that is constructed by the unassigned poles. Here, the subset of PI controller parameters such that the remaining poles are located on the left side of a line $s = -m\sigma$ can be found through this polynomial. It leads to the relative stabilization problem for the polynomial $P_e(s,\sigma,\omega)$; however, it is easily converted to the stability problem over the polynomial $P_e(s - m\sigma,\sigma,\omega)$.

In order to solve the mentioned stability problem above, the polynomial $P_e(j\Omega - m\sigma,\sigma,\omega)$ can be decomposed into its real and imaginary parts and then solved for (Ω,σ) by equating both parts to zero for $\forall\omega^* \in [\omega_{min},\omega_{max}]$. Since solving these equations for every ω^* in the interval is not practical, it is possible to use gridding approach over the parameter ω in order to obtain the solution. After that for the resulting values of $\sigma^* \in [\sigma_{min},\sigma_{max}]$, it is possible to map the achieved (σ^*,ω^*) pairs into the parameter space of PI controller with the help of (12) and (13). Thus, the root boundaries in $K_p - K_i$ plane are found. As a final step, number of the roots, which are located on the right side of the line $s = -m\sigma$ in the resulting regions, can be calculated and the desired parameter region is obtained.

2.2. Pole Region Assignment with PID Controllers

Procedure for the dominant pole region assignment using PID controller is the same with PI controller case. However, in this case, PID controller parameters K_d and K_i are obtained in terms of the parameter K_p apart from the other parameters (such as σ and ω for the $\mathcal{D}_1$ region).

Note that for a PID controller,

$$F(s)=\frac{N_F(s)}{D_F(s)}=\frac{K_d s^2+K_p\,s+K_i}{s} \tag{19}$$

and the followings can be written,

$$N_{F_{Re}}=K_p-K_d\,\sigma-\frac{K_i\,\sigma}{\sigma^2+\omega^2} \tag{20}$$

$$N_{F_{Im}}=K_d\omega-\frac{K_i\,\omega}{\sigma^2+\omega^2} \tag{21}$$

The PID controller parameters K_d and K_i are then obtained as follows with the help of previously defined equations.

$$\begin{pmatrix}K_d(\sigma,\omega)\\ K_i(\sigma,\omega)\end{pmatrix} =\begin{pmatrix}\frac{1}{2\sigma} & \frac{1}{2\omega}\\ \frac{\sigma^2+\omega^2}{2\sigma} & -\frac{\sigma^2+\omega^2}{2\omega}\end{pmatrix}\begin{pmatrix}\frac{N_{G_{Im}}Y-N_{G_{Re}}X}{Z}\\ -\frac{N_{G_{Im}}X+N_{G_{Re}}Y}{Z}\end{pmatrix} +\begin{pmatrix}\frac{1}{2\sigma}\\ \frac{\sigma^2+\omega^2}{2\sigma}\end{pmatrix}K_p \tag{22}$$

After that for a fixed $K_p=k_p^*$, it is again possible to map the $\mathcal{D}$-region to the PID controller parameter space through the functions $K_i(\sigma,\omega)$ and $K_d(\sigma,\omega)$ for the given intervals of the parameters σ and ω. All PID controller set in 3D parameter space can then be found by gridding the K_p parameter. Finally, the sub-region where the remaining poles are located in the desired region is obtained if the same steps are followed as in PI controller case.

3. Numerical Examples

3.1. Example 1 (PI Controller Case)

Consider a fourth order system with the transfer function

$$G(s)=\frac{s-2}{s^4+8s^3+27.5s^2+30s+28} \tag{23}$$

In the closed-loop, it is desired the given system to have damping ratio of $0.6266\le\zeta\le0.826$ and the natural frequency of $0.484\le\omega_n\le0.798$ using a PI controller. Fig. 2 shows the desired dominant pole region.

Firstly, the parametrization of the PI controller parameters over ζ and ω_n is performed using (12) and (13).

$$K_p(\zeta,\omega_n)=-125+\frac{139}{1+\zeta\omega_n+0.25\omega_n^2}+95\zeta\omega_n+(10-40\zeta^2)\omega_n^2+(-4\zeta+8\zeta^3)\omega_n^3 \tag{24}$$

$$K_i(\zeta,\omega_n)=-278+\frac{1112(1+\zeta\omega n)}{4+4\zeta\omega_n+\omega_n^2}+47.5\omega_n^2-20\zeta\omega_n^3+(-1+4\zeta^2)\omega_n^4 \tag{25}$$

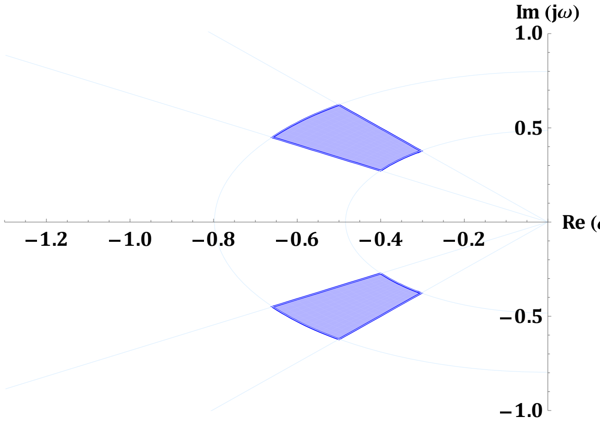

Fig. 2. Desired dominant pole region in s-plane.

It is now possible to map the desired region into parameters space for $\omega_n\in[\omega_{n_{min}},\omega_{n_{max}}]$ and $\zeta\in[\zeta_{min},\zeta_{max}]$. If the parametric plot is drawn in K_p-K_i plane, Fig. 3 is obtained.

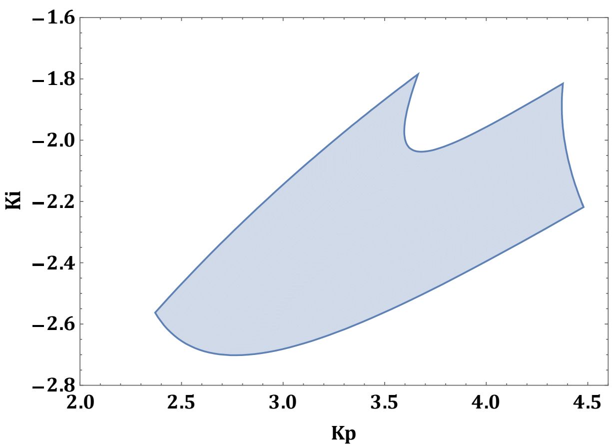

Fig. 3. The corresponding region in K_p-K_i plane.

As long as the PI controller parameters are chosen inside the obtained region, it is guaranteed that dominant poles are assigned to the desired region in s-plane. However, it is also expected the remaining poles to be located away from the dominant pole pair if possible.

Let the remaining poles to be located on the left side of $s=-3\zeta\omega_n$ line that means the dominance factor to be 3 for the closed-loop system. Here, the polynomial $P_e(s,\zeta,\omega_n)$ should be obtained through the closed-loop characteristic polynomial as below.

$$P_c(s,\zeta,\omega_n)=(s^2+2\zeta\omega_n s+\omega_n^2)\,P_e(s,\zeta,\omega_n) \tag{26}$$

The polynomial $\widetilde{P}_e(s,\zeta,\omega_n)=P_e(s-3\zeta\omega_n,\zeta,\omega_n)$ should then be found in order to convert the relative stabilization problem to the stability problem.

$$\widetilde{P}_e(s,\zeta,\omega_n)=s^3+\beta_2 s^2+\beta_1 s+\beta_0 \tag{27}$$

where

$$\beta_0 = -95 + \frac{556}{4 + 4\zeta\omega_n + \omega_n^2} - 42.5\zeta\omega_n + (2 + 112\zeta^2)\omega_n^2 + \zeta(3 - 57\zeta^2)\omega_n^3$$

$$\beta_1 = 27.5 - 64\zeta\omega_n + (-1 + 43\zeta^2)\omega_n^2$$

$$\beta_2 = 8 - 11\zeta\omega_n$$

Let us substitute $s = j\Omega$ and decompose the polynomial $\widetilde{P}_e(s, \zeta, \omega_n)$ into the real and imaginary parts. The root boundaries are found by solving

$$\begin{matrix} Re\left(\widetilde{P}_e(j\Omega, \zeta^*, \omega_n)\right) = 0 \\ Im\left(\widetilde{P}_e(j\Omega, \zeta^*, \omega_n)\right) = 0 \end{matrix} \quad \Longrightarrow \quad \omega_n^* \in [\omega_{n_{min}}, \omega_{n_{max}}] \qquad (28)$$

for a fixed $\zeta^* \in [\zeta_{min}, \zeta_{max}]$ using the gridding approach. Finally, the achieved (ζ^*, ω_n^*) pairs are mapped into the PI controller parameter space. Fig. 4 shows the sub-regions and the number of poles (δ) located on the right side of the line $s = -3\zeta\omega_n$ in those regions. PI controller design process is completed by choosing a (K_p, K_i) pair from the sub-region where $\delta = 0$. Note that only the real root boundary (RRB) exists in this example.

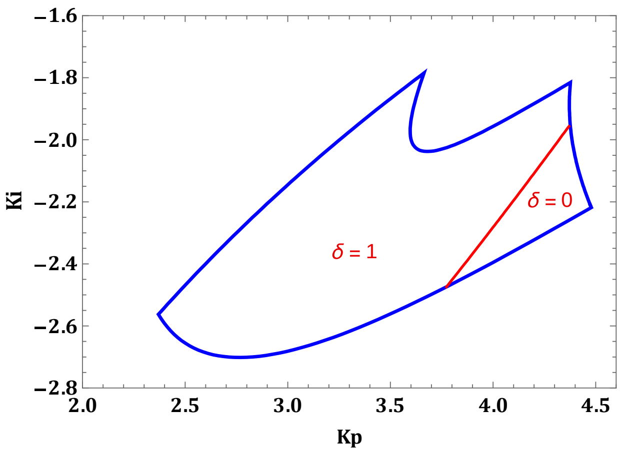

Fig. 4. Obtained sub-region to ensure dominant pole placement.

If the parameters of PI controller are chosen as $K_p = 4.1$ and $K_i = -2.2$ the closed-loop poles are located as in Fig. 5. It is seen from the figures that the dominant poles are assigned in the desired pole region whereas the remaning poles are located on the left side of line $s = -3\zeta\omega_n$. Hence, the dominance factor $m = 3$ is satisfied.

It is worth to note that for higher order systems, it is not always possible to find such a region with PI controller. In this case, the desired closed-loop performance criteria or the dominance factor can be changed.

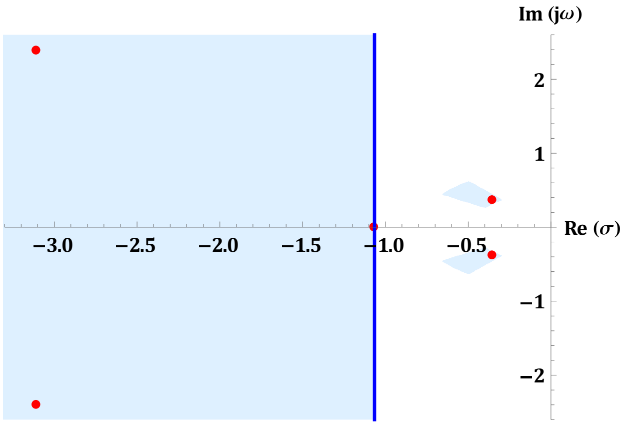

Fig. 5. Closed-loop poles of the system with proposed PI controller.

3.2. Example 2 (PID Controller Case)

Consider a seventh order system with the following transfer function.

$$G(s) = \frac{10}{(s^2+2s+4)(s^2+8s+20)(s+4)^2(s+6)} \qquad (29)$$

Closed-loop performance criteria is bounded by desired overshoot and settling time such that $\zeta \in [0.69, 0.826]$ and $\sigma \in [0.6, 0.9]$. In this case, the desired dominant pole region is illustrated in Fig. 6.

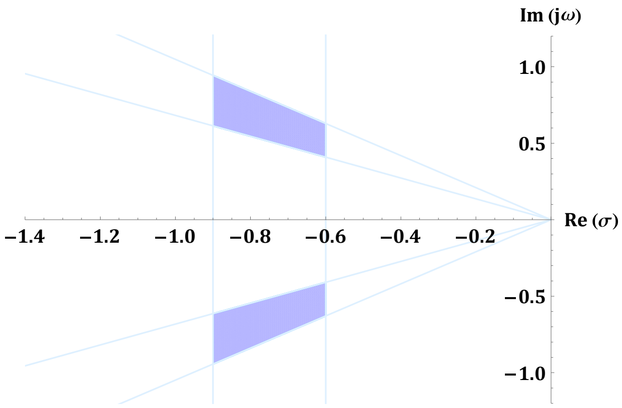

Fig. 6. Desired dominant pole region in s-plane.

Here, the same procedure is used as in previous example; however, the controller to be designed is a PID controller. As explained in section 2, the desired region can be mapped into 2D parameters space (K_i, K_d) for a fixed $K_p = k_p^*$. The final region in 3D space is obtained by gridding the K_p parameter.

For instance, Fig. 7 shows the regions which are mapped from the s-plane for different values of the K_p parameter in $K_d - K_i$ plane. It is also possible to illustrate these regions as a 3D graphic (Fig. 8).

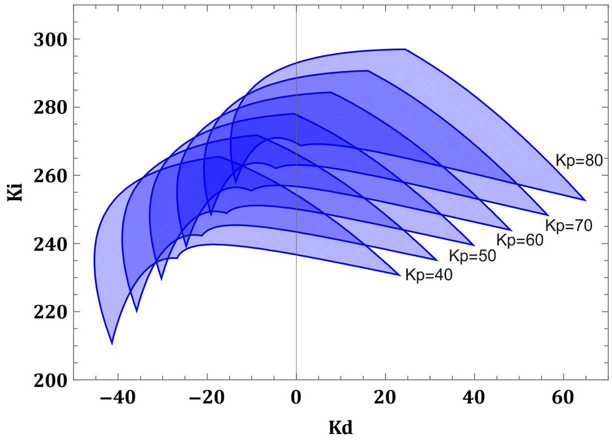

Fig. 7. The corresponding regions in K_d-K_i parameter space.

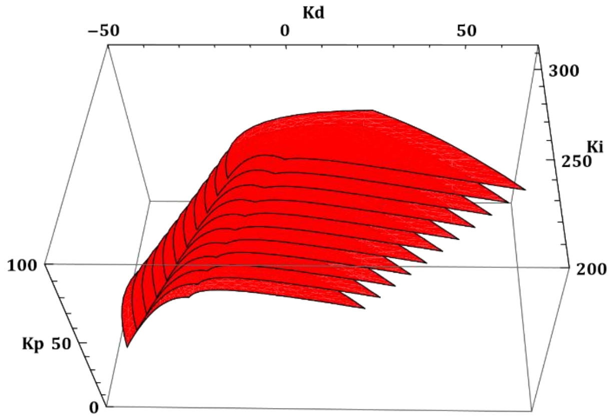

Fig. 8. The corresponding regions in 3D parameter space.

Let the remaining poles to be located on the left side of the line $s = -3\sigma$ that means the dominance factor to be 3 again for the closed-loop system. After the polynomial $\widetilde{P}_e(s,\zeta,\sigma,k_p^*) = P_e(s-3\sigma,\zeta,\sigma,k_p^*)$ is obtained, the root boundaries can be found by equating the real and imaginary parts of the polynomial $\widetilde{P}_e(j\Omega,\zeta,\sigma,k_p^*)$ to zero. The next step is to map the achieved (ζ^*,σ^*) pairs into the $K_d - K_i$ plane. Fig. 9 shows the sub-regions in parameter plane for $K_p = 50$.

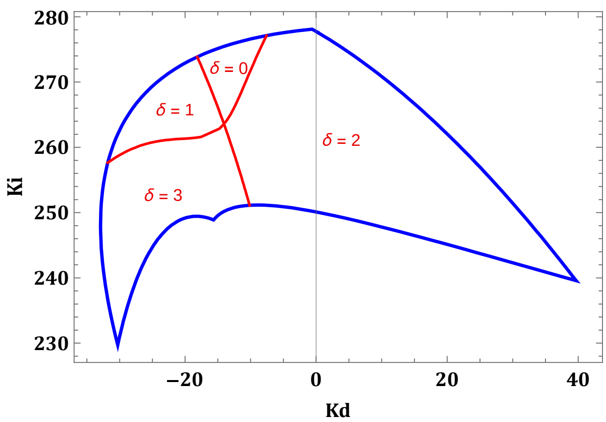

Fig. 9. Sub-regions divided by the root boundaries.

It is seen from the figure that two root boundaries (a real root boundary and a complex root boundary) divide the parameter space into 4 sub-regions. The sub-region where $\delta = 0$ constitutes a solution to our problem. The same calculations are done for $K_p \in [30,80]$ and the regions are obtained as a 3D figure as below.

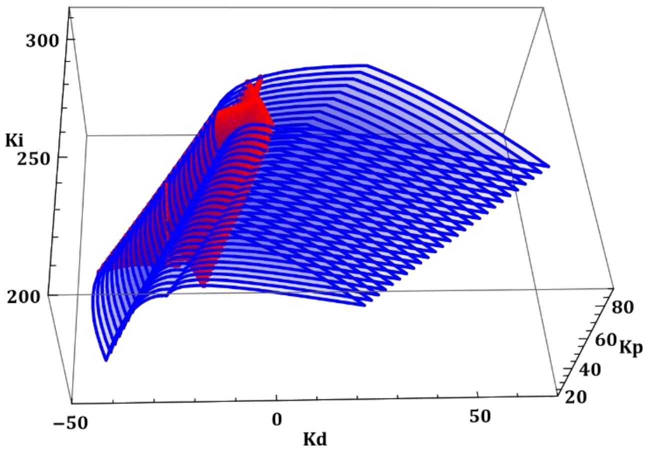

Fig. 10. PID parameters that provide dominant pole region assignment.

Let PID controller parameters be taken as follows.

$$K_p = 50$$
$$K_d = -15$$
$$K_i = 270$$

The closed-loop pole map of the system with designed PID controller is given in Fig. 11. It is clear that dominant pole region assignment is performed successfully through proposed method.

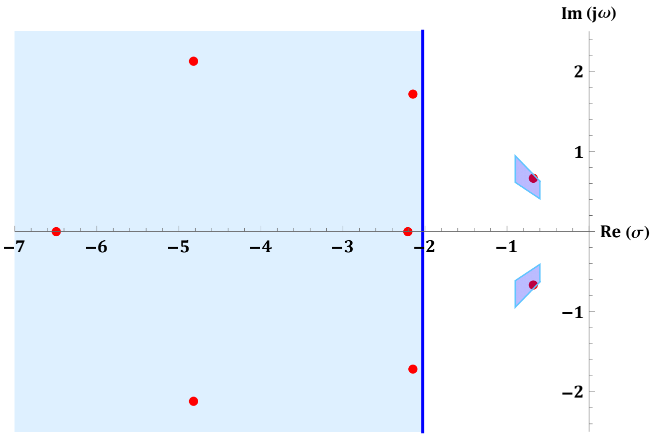

Fig. 11. Closed-loop poles with proposed PID controller.

4. Conclusions

In this study, continuous PI and PID controller design method via the dominant pole region assignment approach is proposed. Thus, it is possible to assign two of the closed-loop poles in a pre-determined region so that the closed-loop system satisfies the time domain characteristics between the minimum and maximum desired values. Furthermore, the non-dominant poles are also located away from the assigned dominant pole pair. Firstly, parametrization of the PI/PID controllers is done and desired dominant pole region is mapped into the controller parameter space. After that the sub-region in which the unassigned poles are located away from the dominant poles is found by calculating the root boundaries. Thus, the PI/PID controller parameters are obtained.

Note that it is not always possible to place remaining poles away from the dominant pole region especially the order of considered system is too high due to the fact that PI/PID controllers can assign only limited number of poles in the closed-loop. Therefore, if the resulting parameter space does not contain a sub-region where $\delta = 0$, the design process should be repeated by changing the performance criteria and/or dominance factor.

5. References

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