Basis Forms of Switching Functions an their Orthogonal Relation

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Abstract

This paper presents extended forms, equations, and relationships of switching function which are derived out of the theoretical handling of switching algebra and the combinational circuit design. Two extended forms of switching functions with the four already existing forms, which will be partly renamed, are presented. Out of combinational circuit design these extended forms, the antivalence of disjunctions A_DF and the equivalence of conjunctions E_CF, are educed and thereby their algebraic expressions are set up. For that, conversion rules which enable the transformation of a disjunction of two variables in equivalence-operation of the same variables and the transformation of a conjunction consisting of two variables in antivalence-operation of the same variables with respect to the extended forms will be lay down. Furthermore, in the case of orthogonal representation of these six function forms relations between them result. That means, by orthogonalization of a corresponding function form it can be easily transformed in an equivalent another function form.

1. Introduction

t_i	t_j	$\bar{t}_i$	$\bar{t}_j$	$t_i \wedge t_j$	$t_i \vee t_j$	$t_i \oplus t_j$	$t_i \odot t_j$
0	0	1	1	0	0	0	1
0	1	1	0	0	1	1	0
1	0	0	1	0	1	1	0
1	1	0	0	1	1	0	1

Table 1. Boolean operations of two terms

The mathematical representation of combinational circuit is done by using of Boolean function respectively of switching function which is defined as the mapping $f(\underline{x}) : \boldsymbol{B}^n \rightarrow \boldsymbol{B}$. $\boldsymbol{B}^n$ is the cartesian product $\boldsymbol{B} \times \boldsymbol{B} \times \cdots \times \boldsymbol{B}$ of $\boldsymbol{B}$ as a set which assigns the values 1 or 0 to each binary vector ($\underline{x} \in \boldsymbol{B}^n$, $\boldsymbol{B} := \{0, 1\}$) [1], [2], [3], [4], [5]. Each entry of the binary vector $\underline{x} = (x_1, x_2, .., x_n)$ represents a Boolean variable which are either negated $\bar{x}_n$ or not-negated x_n. Switching function consists of terms t_n which consists of variables x_n. These terms can be assumed either as conjunctions ($\wedge$, Eq. (1)), disjunctions ($\vee$, Eq. (2)), antivalence term ($\oplus$, Eq. (3)) or equivalence term ($\odot$, Eq. (4)). Their corresponding type of gate are characterized as AND-, OR-, XOR-, XNOR-gate (Tab. 2). Expressions with a fixed simple form are called standard forms of Boolean functions [1]. Each standard form of Boolean functions, which there are four in their number [6], [7] consists of connections of either conjunctions or disjunctions. The disjunction of at least two conjunctions is defined as disjunctive form (Tab. 3, No. I) and is characterized by OR-connection of at least two AND-gates (Fig. 4 a)) in the combinational circuit. In contrast, the conjunction of at least two disjunctions is defined as conjunctive form (Tab. 3, No. II) and is characterized by AND-connection of at least two OR-gates (Fig. 4 b)). In addition, the antivalence form is the antivalence connection of at least two conjunctions (Tab. 3, No. III) and is characterized by XOR-connection of at least two AND-gates (Fig. 4 c)). The equivalence form is defined by the equivalence connection of at least two disjunctions (Tab. 3, No. IV) and is characterized by XNOR-connection of at least two OR-gates (Fig. 4 f)) [8], [9]. The canonical representation of these standard forms called as normal form and denoted as the disjunctive normal form *DNF*, the conjunctive normal form *CNF*, the antivalence normal form *ANF* and the equivalence normal form *ENF*.

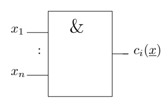
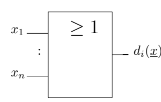
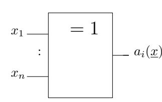
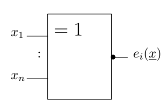
Gate	Formulae
	$c_i(\underline{x}) = \bigwedge_{j=1}^{n} x_j = x_1 \wedge \cdots \wedge x_n$ (1)
	$d_i(\underline{x}) = \bigvee_{j=1}^{n} x_j = x_1 \vee \cdots \vee x_n$ (2)
	$a_i(\underline{x}) = \bigoplus_{j=1}^{n} x_j = x_1 \oplus \cdots \oplus x_n$ (3)
	$e_i(\underline{x}) = \bigodot_{j=1}^{n} x_j = x_1 \odot \cdots \odot x_n$ (4)

Table 2. Six advanced standard forms

2. Extended Standard Forms

The existing four basis forms are deduced from the discrete mathematics and are assigned to the switching algebra due to the isomorphism. Conversely, a corresponding algebraic expression for the switching function can be set up out of their corresponding combinational circuit design, which will be used for mathematical calculation. If the inputs of AND-, OR-, XOR-, XNOR-gate are alternately combined with AND- and OR-gates six different combinal circuits are created. Consequently, extended forms arise [10]. The connection of at least two AND-gates as input of an OR-gate (Fig. 4 a)) corresponds to the disjunctive form *DF* (Tab. 3, No. I).

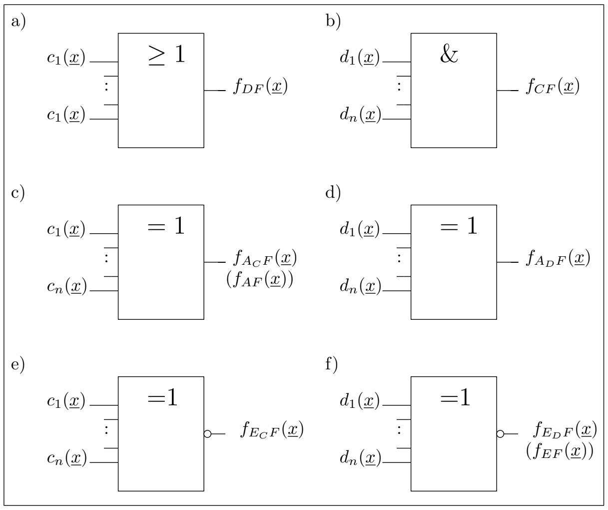


Figure 1. six standard forms as combinational circuit

The connection of OR-gates as input of an OR-gate does not give a new form. It completely originates to an OR-gate. The same applies to the AND connection of AND-gates. The connection of OR-gates as input of AND-gate (Fig. 4b)) corresponds to the conjunctive form *CF* (Tab. 3, No. II). Since NAND- and NOR-gates can be basically represented as the negation of AND- and OR-gates these two forms are not treated in more detail. The antivalence form of conjunctions A_CF (Tab. 3, No. III) appears by XOR connection of AND-gates (Fig. 4c)) and the antivalence form of disjunctions A_DF (Tab. 3, No. V) appears by XOR connection of OR-gates (Fig. 4d)). By XNOR operation of AND-gates (Fig. 4e)) appears the equivalence form of conjunctions E_CF (Tab. 3, No. VI) and by XNOR operation of OR-gates (Fig. 4f)) appears the equivalence form of disjunctions E_DF (Tab. 3, No. IV).

Form	**Formula** (m>1)	**No.**
DF	$f_{DF}(\underline{x}) = \bigvee_{i=1}^{m} c_i(\underline{x})$	I
CF	$f_{CF}(\underline{x}) = \bigwedge_{i=1}^{m} d_i(\underline{x})$	II
A_CF	$f_{A_CF}(\underline{x}) = \bigoplus_{i=1}^{m} c_i(\underline{x})$	III
E_DF	$f_{E_DF}(\underline{x}) = \bigodot_{i=1}^{m} d_i(\underline{x})$	IV
A_DF	$f_{A_DF}(\underline{x}) = \bigoplus_{i=1}^{m} d_i(\underline{x})$	V
E_CF	$f_{E_CF}(\underline{x}) = \bigodot_{i=1}^{m} c_i(\underline{x})$	VI

Table 3. Six advanced standard forms

3. Conversion Rules

Conversion from one form to another equivalent form allows the simpler mathematical treatment in the respective form. Those restructurings allow the equivalent modification of operations. This means, that the reshaped form is equivalent to the original form. Furthermore, this can also enable the reduction of the number of gates of a combinational circuit. Reduction of gates is linked with cost reduction. Rules in respect to the both extended forms will be given by the following equations. Conversions from disjunction to equivalence and from conjunction to antivalence and vice versa can be made by these following expressions in Eq. (5) and Eq. (6). The correctness and their general validity is proved by the use of truth table. The use of truth table illustrates that the function value of the left side (l.s.) is equivalent to the right side (r.s.):

Definition 1. *The conversion of a disjunction of two variables in equivalence-operation is obtained by the following transformation:*

$$x_i \vee x_j = x_i \odot x_j \odot x_i x_j \tag{5}$$

Proof.

x_i	x_j	$x_i \vee x_j$	$x_i \odot x_j$	$x_i x_j$	$x_i \odot x_j \odot x_i x_j$
0	0	0	1	0	0
0	1	1	0	0	1
1	0	1	0	0	1
1	1	1	1	1	1

□

The expression (5) allows the converting of an OR-gate into the join of an AND- and EXNOR-gates. Conversely, this join of two gates can be replaced with one gate and vice versa (Fig. 2).

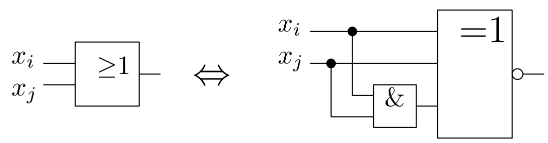


Figure 2. OR-gate into AND- and EXNOR-gates and vice versa

Definition 2. *The conversion of a conjunction of two variables in antivalence-operation is given by the following expression:*

$$x_i x_j = x_i \oplus x_j \oplus (x_i \vee x_j) \tag{6}$$

Proof.

x_i	x_j	$x_i x_j$	$x_i \oplus x_j$	$x_i \vee x_j$	$x_i \oplus x_j \oplus (x_i \vee x_j)$
0	0	0	0	0	0
0	1	0	1	1	0
1	0	0	1	1	0
1	1	1	0	1	1

□

By the use of the expression (6) an AND-gate can be converted into the join of an OR- and EXOR-gates. Conversely, this join of two gates can be replaced with one gate and vice versa (Fig. 3).

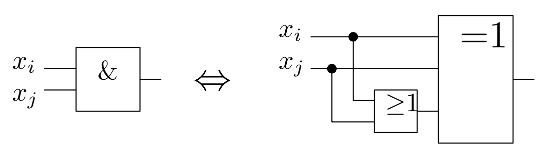


Figure 3. AND-gate into OR- and EXOR-gates and vice versa

Both new expressions can also be used for the proof of the correctness of the relation between the advanced forms when they possess the characteristic of orthogonality which will be shown in the following section.

4. Relation of orthogonal Representations

The orthogonal representation of a basis form, which are of great relevance in solving further Boolean problems, e.g. in the switching algebra [2], [3], is characterized by terms (conjunctions respectively disjunctions) which are disjointed to one another in pairs. Consequently, these conjunctions have no common covering after their logical conjunction, $(c_i(\underline{x}) \wedge c_j(\underline{x}) = 0)$. In contrast, the logical disjunction of these disjunctions results in 1, $(d_i(\underline{x}) \vee d_j(\underline{x}) = 1)$. An orthogonal disjunctive form DF^{orth} is equivalent to the orthogonal antivalence form of conjunctions A_CF^{orth} including the same conjunctions, $DF^{orth} = A_CF^{orth}$ [6], [4]. That means the orthogonalization of a DF facilitates the transformation into an orthogonal A_CF [11], [1], [4], [5], [6]. This characteristic simplifies the handling for further calculations as the Boolean Differential Calculus especially in Ternary-Vector-List-arithmetic [2], [6]. This relationship can be explained well with the following definition out of [5], if both conjunctions $c_i(\underline{x})$ and $c_j(\underline{x})$ are orthogonal to each other:

$$c_i(\underline{x}) \vee c_j(\underline{x}) = c_i(\underline{x}) \oplus c_j(\underline{x}) \oplus \underbrace{(c_i(\underline{x}) \wedge c_j(\underline{x}))}_{=0} \qquad (7)$$

Because of $x_i \oplus 0 = x_i$ it appllies:

$$c_i(\underline{x}) \vee c_j(\underline{x}) = c_i(\underline{x}) \oplus c_j(\underline{x})$$
$$DF^{orth} = A_CF^{orth}$$

Furthermore, an orthogonal conjunctive form CF^{orth} is equivalent to the orthogonal equivalence form of disjunctions E_DF^{orth} including the same disjunctions, $CF^{orth} = E_DF^{orth}$ [6]. By the following definition out of [5] this relationship can be manifested, if both disjunctions $d_i(\underline{x})$ and $d_j(\underline{x})$ are orthogonal to each other:

$$d_i(\underline{x}) \wedge d_j(\underline{x}) = d_i(\underline{x}) \odot d_j(\underline{x}) \odot \underbrace{(d_i(\underline{x}) \vee d_j(\underline{x}))}_{=1} \qquad (8)$$

Because of $x_i \odot 1 = x_i$ it appllies:

$$d_i(\underline{x}) \wedge d_j(\underline{x}) = d_i(\underline{x}) \odot d_j(\underline{x})$$
$$CF^{orth} = E_DF^{orth}$$

Special calculations can be easier solved in another form. For example, building the complement of DF is a complex procedure. However, by orthogonalization DF is transformed in A_CF^{orth} at which the complement by linking with $\oplus$ 1 is subsequently determined, it follows $\overline{A_CF}^{orth}$. By a further orthogonalization of $\overline{A_CF}^{orth}$ the **re**-transforming back to the disjunctive form is gained. Thus, the complement $\overline{DF}$ is calculated. With regard to the orthogonal representation of the new advanced forms following relations apply:

Definition 3. *An orthogonal DF is equivalent to the orthogonal A_CF and equivalent to the complement of the orthogonal E_CF, it applies $DF^{orth} = A_CF^{orth} = \overline{E_CF^{orth}}$:*

$$\bigvee_{i=1}^{n} c_i(\underline{x}) = \bigoplus_{i=1}^{n} c_i(\underline{x}) = \overline{\bigodot_{i=1}^{n} c_i(\underline{x})} = \bigodot_{i=1}^{n} c_i(\underline{x}) \odot 0 \qquad (9)$$

The relation of (9) can be laid down with the following expression which is deduced out of Eq. (6) if both conjunctions $c_i(\underline{x})$ and $c_j(\underline{x})$ are disjunct.

$$c_i(\underline{x}) \vee c_j(\underline{x}) = c_i(\underline{x}) \odot c_j(\underline{x}) \odot \underbrace{(c_i(\underline{x}) \wedge c_j(\underline{x}))}_{=0} \qquad (10)$$

It follows:

$$c_i(\underline{x}) \vee c_j(\underline{x}) = c_i(\underline{x}) \odot c_j(\underline{x}) \odot 0 \qquad (11)$$

The complement of a term can be done by the $\odot$-operation of 0, because it applies: $x_i \odot 0 = \bar{x}_i$. This follows:

$$c_i(\underline{x}) \vee c_j(\underline{x}) = \overline{c_i(\underline{x}) \odot c_j(\underline{x})}$$
$$DF^{orth} = \overline{E_CF^{orth}}$$

Due to this orthogonal relation combinational circuits of DF can be transformed in E_CF as shown in Fig. 4.

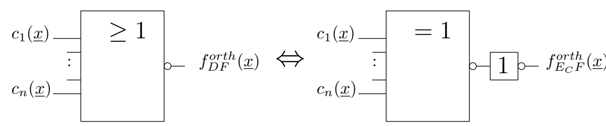


Figure 4. Transformation due to the orthogonal form

Definition 4. *An orthogonal CF is equivalent to the orthogonal E_DF and equivalent to the complement of the orthogonal A_DF, it applies $CF^{orth} = E_DF^{orth} = \overline{A_DF^{orth}}$:*

$$\bigwedge_{i=1}^{n} d_i(\underline{x}) = \bigodot_{i=1}^{n} d_i(\underline{x}) = \overline{\bigoplus_{i=1}^{n} d_i(\underline{x})} = \bigoplus_{i=1}^{n} d_i(\underline{x}) \oplus 1 \qquad (12)$$

The relation of (12) base on the following expression which is deduced out of Eq. (6) if both disjunctions $d_i(\underline{x})$ and $d_j(\underline{x})$ are orthogonal.

$$d_i(\underline{x}) \wedge d_j(\underline{x}) = d_i(\underline{x}) \oplus d_j(\underline{x}) \oplus \underbrace{(d_i(\underline{x}) \vee d_j(\underline{x}))}_{=1} \qquad (13)$$

It follows:

$$d_i(\underline{x}) \wedge d_j(\underline{x}) = d_i(\underline{x}) \oplus d_j(\underline{x}) \oplus 1 \qquad (14)$$

The complement of a term can be done by the $\oplus$-operation of 1, because it applies: $x_i \oplus 1 = \bar{x}_i$. This follows:

$$d_i(\underline{x}) \wedge d_j(\underline{x}) = \overline{d_i(\underline{x}) \oplus d_j(\underline{x})}$$
$$CF^{orth} = \overline{A_DF^{orth}}$$

Due to this orthogonal relation combinational circuits of KF can be transformed in A_DF as shown in Fig. 5.

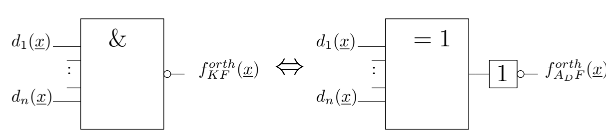


Figure 5. Transformation due to the orthogonal form

Thus, transformations between these six forms can be easily performed by orthogonalization.

5. Conclusion

Two extended forms of switching function - the antivalence form of disjunctions A_DF and the equivalence form of conjunctions E_CF - were investigated in this work. These extended forms were derived from the representation of combinational circuit. Whereby, the other four existing forms were partly renamed in disjunctive form DF, conjunctive form CF, antivalence form of conjunctions A_CF and equivalence form of disjunctions E_DF. Novel equations for the conversion of disjunctions of two variables in equivalence-operation of the same variables and the conversion of conjunctions of two terms in antivalence-operation of the same variables derived for the advanced forms A_DF and E_CF. Their general validity was proven by the use of truth table. Furthermore, these equations for conversion are used for exhibit the correctness of the relations between the extended six basis forms when they have the characteristic of orthogonality. Consequently, relations arise if these six forms are mapped in orthogonal form: an orthogonal DF is equivalent to the orthogonal A_CF and equivalent to the complement of the orthogonal E_CF. thus, it apllies $DF^{\text{orth}} = A_CF^{\text{orth}} = \overline{E_CF^{\text{orth}}}$. Additionally, an orthogonal CF is equivalent to the orthogonal E_DF and equivalent to the complement of the orthogonal A_DF, $CF^{\text{orth}} = E_DF^{\text{orth}} = \overline{A_DF^{\text{orth}}}$.

6. References

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